Assignment 9

This homework is due *Monday*, November 28.

There are total 31 points in this assignment. 28 points is considered 100%. If you go over 28 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 5.1–5.4 in O'Neill.

(1) (5.1.2) Consider the surface $M : z = f(x, y)$, where $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$.
  (a) [2pt] Show that the vectors $u_1 = U_1(0) = (1, 0, 0)_0$ and $u_2 = U_2(0) = (0, 1, 0)_0$ are tangent to M at the origin, and
$$U = \frac{-f_x U_1 - f_y U_2 + U_3}{\sqrt{1 + f_x^2 + f_y^2}}$$
  is a unit normal vector field on M.
  (b) [3pt] Apply the definition of shape operator S to get that $S(u_1) = f_{xx}u_1 + f_{xy}u_2$, $S(2_1) = f_{yx}u_1 + f_{yy}u_2$. (*Hint:* Computations are simplified compared to general Monge surface because of the condition on f above.)

(2) [2pt] (5.3.1) Show that there are no umbilics on a surface with Gaussian curvature $K < 0$, and that when $K \leq 0$, umbilic points are planar. (Note: within this homework, we adopt the textbook definition of umbilics, i.e. allow $k = 0$.)

(3) (5.3.3) (This problem explains the term *mean curvature*.) Prove that
  (a) [2pt] the average value of the normal curvature in *any* two orthogonal directions at a point $p \in M$ is $H(p)$. (*Hint:* Use Euler's formula.)
  (b) [3pt] $H(p) = \frac{1}{2\pi} \int_0^{2\pi} k(\theta) d\theta$. (Thus, mean value of normal curvature is the mean curvature at p.)

(4) (Lemma 5.4.2)
  (a) [2pt] If $\mathbf{x}$ is a patch in $M \subset \mathbb{R}^3$, and U is a unit normal vector field on M, show that
$$L = S(\mathbf{x}_u) \bullet \mathbf{x}_u = U \bullet \mathbf{x}_{uu},$$
$$N = S(\mathbf{x}_v) \bullet \mathbf{x}_v = U \bullet \mathbf{x}_{vv}.$$
  (*Hint:* Use $S(\alpha') \bullet \alpha' = \alpha'' \bullet U$.)

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(b) [2pt] Prove that

$$M = S(\mathbf{x}_v) \bullet \mathbf{x}_u = U \bullet \mathbf{x}_{vu},$$
$$S(\mathbf{x}_u) \bullet \mathbf{x}_v = U \bullet \mathbf{x}_{uv},$$

which means $S(\mathbf{x}_v) \bullet \mathbf{x}_u = S(\mathbf{x}_u) \bullet \mathbf{x}_v$.

(c) [2pt] Given that $S(\mathbf{x}_u) \bullet \mathbf{x}_v = \mathbf{x}_u \bullet S(\mathbf{x}_v)$ by the previous item, prove that for any tangent vectors w_1, w_2, $S(w_1) \bullet w_2 = w_1 \bullet S(w_2)$. (*Hint:* Either use linearity, or decompose w_1, w_2 as in the basis $\mathbf{x}_u, \mathbf{x}_v$.)

(5) [4pt] (5.4.2) For a Monge patch $\mathbf{x}(u, v) = (u, v, f(u, v))$, shot that

$$E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2,$$
$$L = \frac{f_{uu}}{W}, \quad M = \frac{f_{uv}}{W}, \quad N = \frac{f_{vv}}{W},$$

where $W = \sqrt{EG - F^2} = (1 + f_u^2 + f_v^2)^{1/2}$.

Find formulas for K, H. (*Hint:* Use Problem 4. Unit normal vector field is given in Problem 1.)

(6) [2pt] (Example 5.4.3(2)) Find K, H for the saddle surface $z = xy$ by applying the formulas obtained in the previous problem.

(7) [4pt] (5.4.12) Show that a ruled surface $\mathbf{x}(u, v) = \beta(u) + v\delta(u)$ has $N = 0$ and that Gaussian curvature

$$K = \frac{-M^2}{EG - F^2} = \frac{-(\beta' \bullet \delta \times \delta')^2}{W^4},$$

where $W = \|(\beta' \times \delta + v\delta' \times \delta\|$.

(*Hint:* You only need to find E, F, G, M and unit normal U. Use $U = \mathbf{x}_u \times \mathbf{x}_v / \|\mathbf{x}_u \times \mathbf{x}_v\|$.)

(8) [3pt] Curvature of helicoid Consider helicoid M given by a single patch

$$\mathbf{x}(u, v) = (u \sin v, -u \cos v, bv), \quad b \neq 0.$$

Find Gaussian and mean curvature of M. (*Hint:* Either do computations, or explain why answer is the same as in case of "usual" helicoid $(u \cos v, u \sin v, bv)$, computed in textbook.)